Problem 7.4

(a) use filter coeffs: $H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2}$

(b) Use positive powers to extract poles and zeros

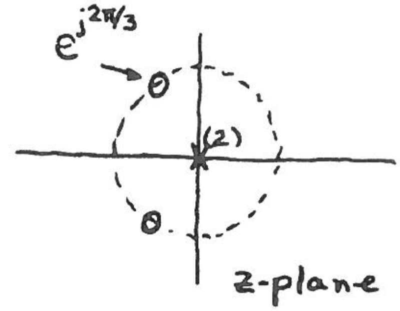
$$H(z) = \frac{1}{z^2}\left(\tfrac{1}{3}z^2 + \tfrac{1}{3}z + \tfrac{1}{3}\right)$$

TWO POLES AT $z=0$

zeros at

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

zeros: $1e^{\pm j2\pi/3}$



(c) $\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$

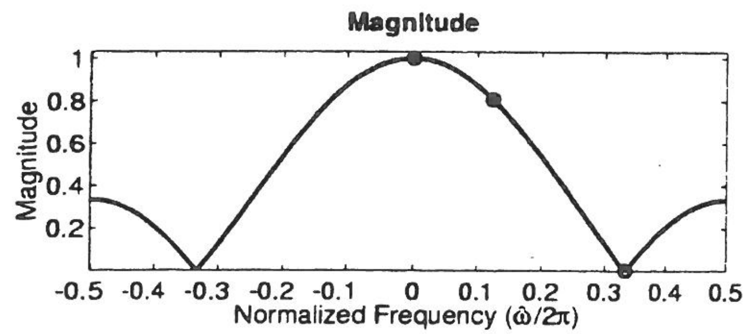
$$= \frac{1}{3} + \frac{1}{3}e^{-j\hat{\omega}} + \frac{1}{3}e^{-j2\hat{\omega}} = \frac{1}{3}e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} + 1 + e^{-j\hat{\omega}}\right)$$

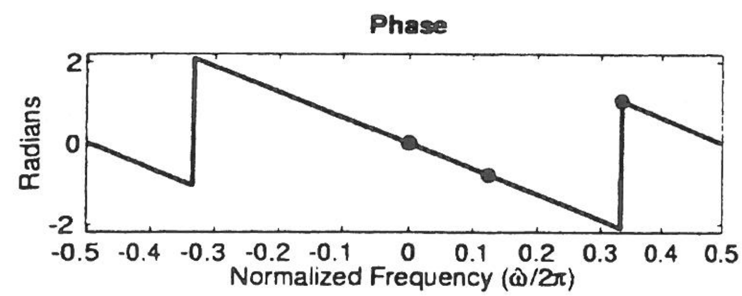
$$= e^{-j\hat{\omega}}\left(\frac{1+2\cos\hat{\omega}}{3}\right)$$

ANOTHER FORMULA:

$$\boxed{\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}\left(\frac{\sin(3\hat{\omega}/2)}{3\sin(\hat{\omega}/2)}\right)}$$

(d) use MATLAB





Problem 7.4 (more)

(e) Use Linearity & Frequency response at $\hat{\omega}=0$, $\hat{\omega}=\pi/4$ and $\hat{\omega}=2\pi/3$. These are marked on the plots of the frequency response.

$$y[n] = 4\mathcal{H}(0) + |\mathcal{H}(\pi/4)|\cos\left(\tfrac{\pi}{4}n - \tfrac{\pi}{4} + \angle\mathcal{H}(\pi/4)\right)$$
$$-3\underbrace{|\mathcal{H}(2\pi/3)|}_{=0}\cos\left(\tfrac{2\pi}{3}n + \angle\mathcal{H}(2\pi/3)\right)$$

$$\mathcal{H}(0) = \tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3} = 1$$

$$\mathcal{H}(\pi/4) = e^{-j\pi/4}(1+2\sqrt{2}/2)/3 = \frac{1+\sqrt{2}}{3}e^{-j\pi/4} = 0.8047e^{-j\pi/4}$$

$$\mathcal{H}(2\pi/3) = 0 \quad \text{because } H(z)=0 \text{ at } z=e^{\pm j2\pi/3}$$

$$\therefore\ y[n] = 4 + 0.8047\cos\left(\tfrac{\pi}{4}n - \tfrac{\pi}{2}\right)$$

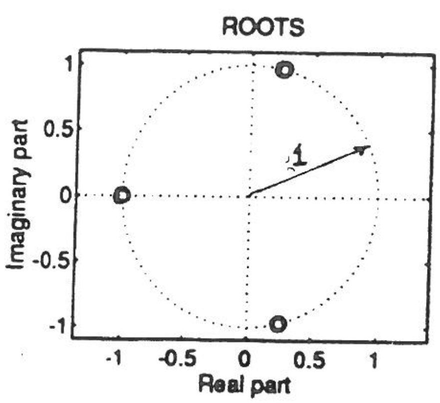
Problem 7.7

$$P(z) = 1 + \tfrac{1}{2}z^{-1} + \tfrac{1}{2}z^{-2} + z^{-3}$$

NOTE $P(-1) = 1 - \tfrac{1}{2} + \tfrac{1}{2} - 1 = 0 \Rightarrow$ ROOT @ $z = -1$

$$\Rightarrow P(z) = (1 + z^{-1})(1 - \tfrac{1}{2}z^{-1} + z^{-2})$$

USE QUADRATIC FORMULA ON THIS PART



$$\frac{\frac{1}{2} \pm \sqrt{1/4 - 4}}{2}$$

$$= \underbrace{\frac{1}{4} \pm j\frac{\sqrt{15}}{4}}$$

MAG OF THESE ROOTS IS EXACTLY ONE.

Problem 7.9

(a) A 4-point moving average is

$$y[n] = \frac{1}{4}\left(x[n] + x[n-1] + x[n-2] + x[n-3]\right)$$

$$H_1(z) = H_2(z) = \frac{1}{4} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3}$$

$$H(z) = H_1(z)H_2(z) = \frac{1}{16}\left(1 + z^{-1} + z^{-2} + z^{-3}\right)^2$$

(b) Find the poles and zeros of $H_1(z)$, then "double" them. Switch to positive powers of z.

$$H_1(z) = \frac{z^3+z^2+z+1}{4z^3}$$

Numerator factors:

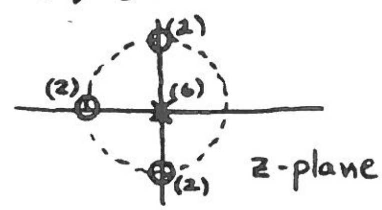
$$(z+1)(z^2+1)$$
$$= (z+1)(z+j)(z-j)$$

3 poles at $z=0$

$\Rightarrow$ Zeros at $z = -1, -j, +j$

(c) For the freq. response

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$



$$H(e^{j\hat{\omega}}) = \frac{1}{16}\Big(\underbrace{1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}}}\Big)^2$$

This can be reduced to a Dirichlet form.

$$H(e^{j\hat{\omega}}) = \frac{1}{16}\left(\frac{\sin(4\hat{\omega}/2)}{\sin(\hat{\omega}/2)}\, e^{-j3\hat{\omega}/2}\right)^2$$

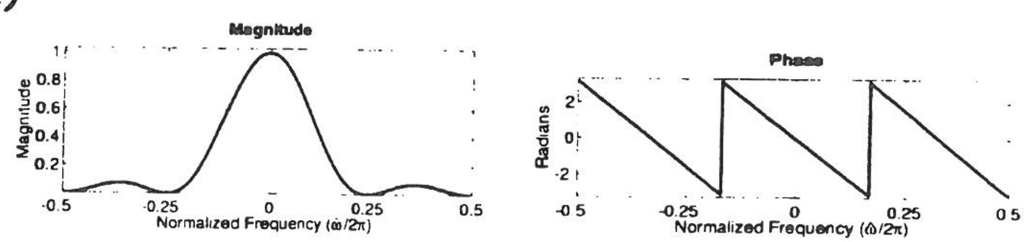
$$= e^{-j3\hat{\omega}}\left(\frac{\sin(2\hat{\omega})}{4\sin(\hat{\omega}/2)}\right)^2$$

phase

Numerator is zero for $2\hat{\omega} = \pi k$ $\Rightarrow$ $\hat{\omega} = \pi k/2$

At $\hat{\omega}=0$, denominator is also zero

(d)



Problem 7.9 (more)

(e)

$$H(z) = \frac{1}{16}\left(1 + z^{-1} + z^{-2} + z^{-3}\right)^2$$

$$= \frac{1}{16}\left(1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}\right)$$

Invert term by term

$$h[n] = \frac{1}{16}\delta[n] + \frac{1}{8}\delta[n-1] + \frac{3}{16}\delta[n-2] + \frac{1}{4}\delta[n-3] + \frac{3}{16}\delta[n-4]$$

$$+ \frac{1}{8}\delta[n-5] + \frac{1}{16}\delta[n-6]$$

Problem 7.15

$$x(t) = 4 + \cos(250\pi t - \pi/4) - 3\cos\left(\frac{2000\pi}{3} t\right)$$

with $f_s = 1000$

$$x[n] = x(t)\Big|_{t=n/f_s} = 4 + \cos\left(\frac{\pi n}{4} - \frac{\pi}{4}\right) - 3\cos\left(\frac{2\pi}{3} n\right).$$

Now, run $x[n]$ through the filter $H(z)$.

To do so, we need frequency response at $\hat{\omega} = 0,\ \pi/4,\ 2\pi/3$

$$H(e^{j\hat{\omega}}) = \frac{1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}}{3}$$

$$H(e^{j0}) = \frac{1+1+1}{3} = 1$$

$$H(e^{j\pi/4}) = \frac{1}{3}\left(1 + e^{-j\pi/4} + e^{-j\pi/2}\right) = \frac{1}{3}\left(1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - j\right)$$

$$= \frac{2+\sqrt{2}}{6} - j\,\frac{2+\sqrt{2}}{6} = 0.569 - j0.569 = .8047\, e^{-j\pi/4}$$

$$H(e^{j2\pi/3}) = \frac{1}{3}\left(1 + e^{-j2\pi/3} + e^{-j4\pi/3}\right) = 0$$

So, the output of the digital filter is:

$$y[n] = 4 + 0.8047\cos\left(\frac{\pi}{4}n - \frac{\pi}{2}\right) + 0$$

Now convert back to analog

$$n \longrightarrow f_s t = 1000t$$

$$y(t) = 4 + 0.8047\cos(250\pi t - \pi/2)$$

$$\text{or} \quad = 4 + 0.8047\sin(250\pi t)$$